

Transformation laws of the components of classical and quantum fields and Heisenberg relations

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Abstract

The paper recalls and point to the origin of the transformation laws of the components of classical and quantum fields. They are considered from the "standard" and fibre bundle point of view. The results are applied to the derivation of the Heisenberg relations in quite general setting, in particular, in the fibre bundle approach. All conclusions are illustrated in a case of transformations induced by the Poincaré group.

1. Introduction

The components of physical fields with respect to a reference frame change when this frame changes. The correspondence between the components of a field with respect to two frames of reference is known as a transformation law of that field. These laws are important characteristics of the physical fields; e.g. via them one can make a distinction between vector and spinor fields.

This paper recalls and shows the origin of the transformation laws of the physical fields in the classical case (section 2) and in the quantum one (section 3). A conclusion is made that in the transformations of the classical fields are involved passive coordinate transformations while in the quantum case are presented active coordinate transformations, i.e. diffeomorphisms associated with coordinate transformations. In section 4 are considered the transformation laws of the fields from fibre bundle point of view in which the fields are described as sections of suitable vector bundles. In this setting in the transformation laws of classical as well as quantum fields are involved only passive coordinate transformations. Special attention is paid to transformations induced by the Poincaré group.

The results obtained are applied in section 5 to derivation of the so-called Heisenberg relations in quite general setting.¹ The investigation start with transformations induced by the Poincaré group; in particular, some well known results are reproduced. The consideration of internal transformations leads to Heisenberg relations concerning different charges. The Heisenberg relations are derived also in the general setting when the transformations are induced by three representations of a given group. At last, we look on the Heisenberg relations from fibre bundle view point.

Section 6 closes the work with consideration of some problems concerning the observability and measurability of the reference frames and the components of the physical fields in them.

2. Transformation laws of classical fields

Let M be the Minkowski spacetime model of special relativity. A (classical) field φ describing (some property of) a physical system is a mapping $\varphi: M \rightarrow V$ where V is a real or complex vector space of finite dimension.^{2, 3} Usually V is a space in which a representation of the Poincaré group (in particular, its subgroup the Lorentz group) acts (*vide infra*). The field φ is equivalently described via its components φ^i , $i = 1, \dots, \dim V$, in some frame $\{e_i\}$ ⁴ over M in V , i.e.⁵

$$\varphi^i: M \rightarrow \mathbb{K} \quad \varphi(x) =: \varphi^i(x)e_i(x) \quad x \in M, \quad (2.1)$$

where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, depending on whether a real or complex field is considered, and we write $\varphi = \varphi^i e_i$ meaning that $\varphi^i e_i: x \mapsto \varphi^i(x)e_i(x) = \varphi(x)$. Often a field φ is depicted as a vector-column $(\varphi^1, \dots, \varphi^{\dim V})^\top$, \top being the matrix transposition sign, formed from the \mathbb{K} -valued functions φ^i , called the *components* of φ . One should be aware of the fact that in the (physical) literature as components of φ are referred the (local, if M is a general manifold) representations

$$\varphi_u^i := \varphi^i \circ u^{-1}: \mathbb{R}^4 \rightarrow \mathbb{K} \quad (2.2)$$

of φ^i in some (local) chart (U, u) of M with $u: M \supseteq U \rightarrow \mathbb{R}^4$. However, since as a set M coincides with \mathbb{R}^4 and one (implicitly) thinks of u as the identity mapping of \mathbb{R}^4 , practically

¹ Section 5 is an extended version of [1].

² In a case of Newtonian physics M must be the Newtonian 3-dimensional absolute space.

³ It seems there are not real classical systems described via infinite dimensional vector spaces. But in the fibre bundle quantum case infinitely dimensional spaces may arise naturally.

⁴ That is $e_i: x \rightarrow e_i(x) \in V$ and $\{e_i(x)\}$ is a basis of V .

⁵ From here on in our text the Latin indices i, j, k, \dots run from 1 to the dimension $\dim V$ of V and the summation convention over indices repeated on different levels is assumed over the whole range of their values.

everywhere in the literature the authors write φ^i and talk about it having in mind and dealing, in fact, with φ_u^i .

The transformation laws of fields (more precisely, of their components) are quite important their characteristics. Before formulating them explicitly, some remarks have to be made.

Mathematically the components φ^i of a field φ are the functions $\varphi^i: M \rightarrow \mathbb{K}$ as defined above. They are defined relatively to some frame $\{e_i\}$ over M . Similarly, the coordinates x^μ , $\mu = 0, 1, 2, 3$,⁶ of a spacetime point $x \in M$ are the numbers $x^\mu \in \mathbb{R}$ defined by the relation $u(x) =: (x^0, x^1, x^2, x^3) =: \mathbf{x}$. They are given with respect to a coordinate system $\{u^\mu\}$ with $u^\mu: x \mapsto u^\mu(x) =: x^\mu$.

Physically the components of a field $\varphi: M \rightarrow V$ or the coordinates of a point $x \in M$ are defined with respect to a given *reference frame*. Among other things, the last concept includes a way of measuring the spacetime coordinates of points and the components of all fields concerning particular problem (system)⁷; in our case the functions φ^i (or their values $\varphi^i(x) = \varphi_u^i(u(x)) \in \mathbb{K}$) and the numbers x^μ . So, in the particular situation, a frame of reference is mathematically described via a pair $(u, \{e_i\})$ of a coordinate system⁸ u of M and frame $\{e_i\}$ over M in V . We write u at first position in $(u, \{e_i\})$ as generally the frames $\{e_i\}$ may depend on u (see below subsection 4.2).

Suppose $(u, \{e_i\})$ and $(u', \{e'_i\})$ represent two reference frames and

$$e'_i(x) = A_i^j(x)e_j(x) \quad (2.3)$$

for some non-degenerate matrix-valued function $A = [A_i^j]$. We admit that e_i (resp. e'_i) may depend on u (resp. u'), so A may depend on u and u' .⁹ A *transformation law* of a field φ (relative to $(u, \{e_i\})$ and $(u', \{e'_i\})$) is called the correspondence $\varphi_u^i \mapsto \varphi_{u'}^i$. Writing the decompositions $\varphi(x) = \varphi^i(x)e_i(x) = \varphi'^i(x)e'_i(x)$ and using $\varphi_u^i := \varphi^i \circ u^{-1}$, $\varphi_{u'}^i := \varphi'^i \circ u'^{-1}$ and $e'_i = A_i^j e_j$, we get the explicit form of the transformed field components (relative to $(u', \{e'_i\})$):

$$\varphi'^i(x) = (A^{-1}(x))^i_j \varphi^j(x) \quad (2.4)$$

$$\varphi_{u'}^i(r) = (A^{-1}(u'^{-1}(r)))^i_j \varphi_u^j((u \circ u'^{-1})(r)) \quad (2.4')$$

for any $x \in U \cap U' \subseteq M$ and $r \in u(U) \cap u'(U') \subseteq \mathbb{R}^4$.

Usually, in the physical literature, this formula is written in the following more concrete form. Let u and u' be *linear* and u' be obtained from u by a Poincaré transformation, viz.

$$u'(x) = \Lambda u(x) + a \quad (2.5)$$

with fixed $a \in \mathbb{R}^4$ and Λ being a Lorentz transformation (i.e. the matrix of a 4-rotation). Let in V acts a representation D of the Lorentz (Poincaré) group under which φ transforms, i.e. $u \mapsto u'$ implies

$$A^{-1}(x) = D(\Lambda, a) \quad (2.6)$$

for all $x \in M$ with $D(\Lambda, a) = [D_j^i(\Lambda, a)]$ being the matrix of $D(\Lambda, a): V \rightarrow V$, corresponding to (2.5), in $\{e_i\}$. Then equation (2.4') reduces to

$$\varphi_{u'}^i(r) = D_j^i(\Lambda, a) \varphi_u^j(\Lambda^{-1}(r - a)), \quad (2.7)$$

⁶ In this paper the Greek indices λ, μ, ν, \dots run from 0 to 3 = dim M - 1 and refer to the Minkowski space M .

⁷ See section 6 for some details concerning measurability of fields components.

⁸ As $u(x) = (u^0(x), \dots, u^3(x)) \in \mathbb{R}^4$, we identify the coordinate homeomorphism $u: U \rightarrow \mathbb{R}^4$ with $(u^0, \dots, u^3): x \mapsto (u^0(x), \dots, u^3(x))$ and, using some freedom of the language, we call u also coordinate system regardless that the last term means $\{u^\mu\}$ or (u^0, \dots, u^3) . Note also that $u^\mu = r^\mu \circ u$ where $\{r^\mu\}$ is the standard Cartesian coordinates system on \mathbb{R}^4 , i.e. $r^\mu: (x^0, \dots, x^3) \rightarrow x^\mu$.

⁹ More precisely, A may depend on the transformation $u \mapsto u'$.

where we have used the first of the following simple corollaries from (2.5):

$$(u \circ u'^{-1})(r) = \Lambda^{-1}(r - a) \quad (2.8a)$$

$$(u' \circ u^{-1})(r) = \Lambda r + a \quad (2.8b)$$

$$(u^{-1} \circ u')(x) = u^{-1}(\Lambda u(x) + a) \quad (2.8c)$$

$$(u'^{-1} \circ u)(x) = u^{-1}(\Lambda^{-1}(u(x) - a)). \quad (2.8d)$$

If we identify $x \in M$ with $u(x) = (x^0, \dots, x^3) \in \mathbb{R}^4$ and omit the indices u and u' , equation (2.7) takes the familiar form [2, § 2]

$$\varphi'^i(x) = D_j^i(\Lambda, a) \varphi^j(\Lambda^{-1}(x - a)). \quad (2.9)$$

3. Transformation laws of quantum fields

After quantization a classical field φ transforms into a vector operator-valued distribution (generalized function) $\varphi = \sum_{i=1}^{\dim V} (0, \dots, 0, \varphi_i, 0, \dots, 0)$, where φ_i is usual operator-valued distribution which sits into the i^{th} position. The action of φ on a vectorial test function $\mathbf{f} = (f^1, \dots, f^{\dim V})$, $f^i: \mathbb{R}^4 \rightarrow \mathbb{K}$, is often written symbolically as

$$\varphi(\mathbf{f}) =: \int \sum_i \varphi_i(r) f^i(r) d^4r \quad (3.1)$$

where the integration is over \mathbb{R}^4 and $\varphi_i(r)$ are operators in the system's Hilbert space of states that are treated as operator-valued components of the (nonsmeared) quantum field.¹⁰

Let O and O' be two observers (reference frames). Since the functional φ describes a quantum field irrespectively of any observers, the transition $O \mapsto O'$ should imply $\varphi \mapsto \varphi'$ but the functions on which φ acts can be observer-dependent, i.e. $\mathbf{f} \mapsto \mathbf{f}'$, where \mathbf{f}' (or, more precisely, $\varphi(\mathbf{f}')$) may be determined as follows. Suppose U is an (invertible, linear, and, possibly, unitary) operator representing the change $O \mapsto O'$ of the state vectors of the system (the fields, in our case)¹¹, i.e. if X is a state vector relative to O , the vector $U(X) = X'$ represents the same state with respect to O' . As $\varphi(\mathbf{f}')$ plays with respect to O' the same role as $\varphi(\mathbf{f})$ relative to O , the vector $U((\varphi(\mathbf{f}))(X))$, representing $(\varphi(\mathbf{f}))(X)$ relative to O' , and $(\varphi(\mathbf{f}'))(U(X))$, representing the action of $\varphi(\mathbf{f}')$ on the transformed vector $U(X)$, should be equal, $U((\varphi(\mathbf{f}))(X)) = (\varphi(\mathbf{f}'))(U(X))$. Therefore, we have¹²

$$\varphi(\mathbf{f}') = U \circ \varphi(\mathbf{f}) \circ U^{-1}. \quad (3.2)$$

The transition $\varphi(\mathbf{f}) \mapsto \varphi(\mathbf{f}')$ can effectively and equivalently be describe by admitting that the test function f remains unchanged, while the field φ transforms into φ' , i.e.

$$\varphi'(f) = \varphi(f). \quad (3.3)$$

From here and (3.2) one immediately gets

$$\varphi' = U \circ \varphi \circ U^{-1} \quad (3.4)$$

or equivalently

$$\varphi'_i(r) = U \circ \varphi_i(r) \circ U^{-1}. \quad (3.5)$$

¹⁰ For a rigorous description of quantum fields, see [3, 4].

¹¹ We use one and the same letter U to denote the just mentioned operator and a neighborhood in the Minkowski spacetime and hope that this will not lead to misunderstandings further in this paper.

¹² The above discussion is not rigorous and should be considered only as a motivation. In fact, equation (3.2) below must be postulated. Similarly, equation (3.5) can be considered as a definition of φ'_i .

If we have three observers O_1 , O_2 and O_3 and U_{ab} , $a, b = 1, 2, 3$, maps the state vectors relative to O_b into ones relative to O_a , then

$$U_{aa} = \text{id} \quad (3.6)$$

$$U_{ab} \circ U_{bc} = U_{ac}, \quad (3.7)$$

where id is the identity mapping, and hence the set $\{U: U \text{ maps state vectors between two observers}\}$ has a structure of a “partial” monoid (groupoid with identity/unit element whose operation (multiplication) is not defined on all elements). In a case of inertial observers (connected via Poincaré transformations) the mappings U form a representation of the Poincaré group and therefore $\{U\}$ has a group structure.

Now we want to find the analogue of (2.4') (in particular, of (2.9)) in the quantum case, i.e. we would like to derive the transformation law of the *components* of a quantum field φ .

Treating f^j as components of a classical field and writing f_u^j for f^j , we, due to (2.4'), have

$$f_{u'}^i(r) = (A^{-1}(u'^{-1}(r)))^i_j f_u^j((u \circ u'^{-1})(r)) \quad (3.8)$$

which in a case of Poincaré transformation reduces to

$$f_{u'}^i(r) = D_j^i(\Lambda, a) f_u^j(\Lambda^{-1}(r - a)). \quad (3.9)$$

If we identify x and $r = u(x)$ and omit the subscripts u and u' , then the last equation reads

$$f'^i(x) = D_j^i(\Lambda, a) f^j(\Lambda^{-1}(x - a)) \quad (3.10)$$

which coincides with [3, p. 249, eq. 9.5] up to notation.

The above-said implies that φ (resp. \mathbf{f}) can be regarded as observer independent (resp. dependent), i.e. with respect to any observer O its components $\varphi_i(r)$ (resp. $f_u^i(r)$) are independent of (resp. dependent on) the coordinate system u and frame $\{e_i\}$ associated with O (resp. and transforming according to (2.4') when O is replace with O'). So, we have ¹³

$$\begin{aligned} \varphi(\mathbf{f}') &= \int \varphi_i(r) f_{u'}^i(r) d^4r = \int \varphi_i(r) (A^{-1}(u'^{-1}(r)))^i_j f_u^j((u \circ u'^{-1})(r)) d^4r \\ &= \int \varphi_i((u' \circ u^{-1})(q)) (A^{-1}(u^{-1}(q)))^i_j f_u^j(q) \frac{\partial(u' \circ u^{-1})(q)}{\partial q} d^4q, \end{aligned} \quad (3.11)$$

where (2.4') was used, the variable r has been changed to $q = (u \circ u'^{-1})(r)$, and $\frac{\partial(u' \circ u^{-1})(q)}{\partial q}$ is (the symbolic notation for) the Jacobian of the change $r \mapsto q$.

Now, writing $\varphi'(\mathbf{f}) = \int \sum_i \varphi'_{u',i}(q) f_u^i(q) d^4q$ (see (3.1)), from (3.3) and (3.11), we get

$$\varphi'_{u',i}(r) = \frac{\partial(u' \circ u^{-1})(r)}{\partial r} (A^{-1}(u^{-1}(r)))^j_i \varphi_{u,j}((u' \circ u^{-1})(r)). \quad (3.12)$$

In particular, if u and u' are linear, O and O' are connected via (proper) Poincaré transformation, $u'(x) = \Lambda u(x) + a$ with $x \in M$, then (3.12) reduces to

$$\varphi'_{u',i}(r) = D_i^j(\Lambda, a) \varphi_{u,j}(\Lambda r + a) \quad (3.13)$$

due to (2.8b). Here we have used that the Jacobian of $u \mapsto u'$ equals to one (for proper transformations) and $D(\Lambda, a)$ is the same matrix which appears in (2.9). If $r = u(x)$ is

¹³ For the correctness of the integrals (over \mathbb{R}^4) below, e.g. in (3.11), we have to assume that the neighborhoods U and U' are such that $u(U \cap U') = u'(U' \cap U) = \mathbb{R}^4$ which is equivalent to $u(U) = u'(U') = \mathbb{R}^4$. However, this assumption is not needed for the equations in which integrals are not involved in which case we require $r \in u(U \cap U') \cap u'(U' \cap U)$.

identified with x and the subscripts u and u' are omitted, this formula reduces to the usual one (see, e.g., [2, § 10] and [5, § 68])

$$\varphi'_i(x) = D_i^j(\Lambda, a)\varphi_j(\Lambda x + a). \quad (3.14)$$

Notice, the classical fields φ^i transform according to (2.9) with $\mathbf{D}(\Lambda, a)$, while the quantum ones transform via (3.14) in which the transposed matrix $\mathbf{D}^\top(\Lambda, a)$ of $\mathbf{D}(\Lambda, a)$ is utilized.

The above discussion can be summarized in the following three equations

$$\varphi'_i(r) = U \circ \varphi_i(r) \circ U^{-1} \quad (3.15)$$

$$\varphi'_{u',i}(r) = \frac{\partial(u' \circ u^{-1})(r)}{\partial r} (A^{-1}(u^{-1}(r)))^j_i \varphi_{u,j}((u' \circ u^{-1})(r)) \quad (3.16)$$

$$U \circ \varphi_{u,i}(r) \circ U^{-1} = \frac{\partial(u' \circ u^{-1})(r)}{\partial r} (A^{-1}(u^{-1}(r)))^j_i \varphi_{u,j}((u' \circ u^{-1})(r)) \quad (3.17)$$

the last of which is a consequence of the preceding two ones. At this point we have to say that equation (3.15) is a consequence of the hypotheses/assumptions (3.2) and (3.3), while (3.16) is a result of the hypotheses/assumptions (3.3) and (3.8). If all of the equations (3.2), (3.3) and (3.8) hold, we can write the chain equality

$$\varphi'_{u',i}(r) = U \circ \varphi_{u,i}(r) \circ U^{-1} = \frac{\partial(u' \circ u^{-1})(r)}{\partial r} (A^{-1}(u^{-1}(r)))^j_i \varphi_{u,j}((u' \circ u^{-1})(r)) \quad (3.18)$$

which, in the above special case of Poincaré transformation, reduces to the known result [5, eq. (11.67)]

$$\varphi'_i(x) = U(\Lambda, a) \circ \varphi_i(x) \circ U^{-1}(\Lambda, a) = D_i^j(\Lambda, a)\varphi_j(\Lambda x + a), \quad (3.19)$$

where x is identified with $u(x)$, the subscripts u and u' are omitted, and $U(\Lambda, a)$ is the element corresponding to (2.5) via a representation of the Poincaré group on the space of state vectors.¹⁴

If we set

$$\varphi_{u,i} = \varphi_i \circ u^{-1} \quad f_u^i = f^i \circ u^{-1} \quad (3.20)$$

and $x = u'^{-1}(r) \in M$, then we can rewrite (3.8) and (3.18) respectively as

$$f'^i(x) = (A^{-1}(x))^i_j f^j(x) \quad (3.21)$$

$$\begin{aligned} \varphi'_i(x) &= U \circ \varphi_i((u^{-1} \circ u')(x)) \circ U^{-1} \\ &= \frac{\partial(u' \circ u^{-1})(r)}{\partial r} \Big|_{r=u'(x)} (A^{-1}((u^{-1} \circ u')(x)))^j_i \varphi_j((u^{-1} \circ u')^2(x)). \end{aligned} \quad (3.22)$$

In the special case of a Poincaré transformation, the last two equations reduce to (see (2.6) and (2.8b))

$$f'^i(x) = D_j^i(\Lambda, a)f^j(x) \quad (3.23)$$

$$\begin{aligned} \varphi'_i(x) &= U(\Lambda, a) \circ \varphi_i(u^{-1}(\Lambda u(x) + a)) \circ U^{-1}(\Lambda, a) \\ &= D_i^j(\Lambda, a)\varphi_j(u^{-1}(\Lambda u(x) + a)). \end{aligned} \quad (3.24)$$

In this way we see a very essential difference between the transformation laws of classical fields, like f^i , and quantum ones, like φ_i , when the frame of reference is changed:

¹⁴ The equation (3.19), which holds in relativistic quantum mechanics too, can be derived also under the following assumptions (cf. [5, §68]): (i) the state vectors do not change under a (passive) Poincaré transformations, $X(x) = X(u^{-1}(r)) = X(u'^{-1}(r'))$ for $r = u(x)$, $r' = u'(x)$ and $r' = \Lambda r + a$; (ii) there is a unitary operator $U(\Lambda, a)$ such that $X'(x) = U(\Lambda, a)(X(x))$; (iii) there is a non-degenerate matrix $D(\Lambda, a) = [D_i^j(\Lambda, a)]$ such that $\varphi_i \circ u'^{-1}(r) = \varphi'_{u',i}(r') = D_i^j(\Lambda, a)\varphi_j((u' \circ u^{-1})(r))$; (iv) the scalar products in system's Hilbert space of states are invariant, i.e. $\langle X'(u'^{-1}(r')) | \varphi'_{u',i}(r') \rangle \langle Y'(u'^{-1}(r')) \rangle = \langle X(u^{-1}(r)) | \varphi_{u,i}(r) \rangle \langle Y(u^{-1}(r)) \rangle$. In fact, these suppositions immediately imply $U^{-1}(\Lambda, a) \circ D_i^j(\Lambda, a)\varphi_{u,j}(r) \circ U(\Lambda, a) = \varphi_{u,i}(r)$, i.e. $U^{-1}(\Lambda, a) \circ \varphi_{u,i}(r) \circ U(\Lambda, a) = D_i^j(\Lambda, a)\varphi_{u,j}(r) = \varphi'_{u',i}(r)$. Now, identifying x with $r = u(x)$ and writing φ_i for $\varphi_{u,i}$ and φ'_i for $\varphi'_{u',i}$, we obtain (3.19).

- The classical fields transform according to *passive* coordinate transformations, i.e. in the both sides of (3.21) (or (3.8)) are involved quantities evaluated *at one and the same spacetime point* x .
- Contrary to the previous observation, the quantum fields transform according to *active* coordinate transformations which means that in the both sides of (3.22) (cf. (3.12)) are involved quantities evaluated *at different spacetime points*, viz. x , $(u^{-1} \circ u')(x)$ and $(u^{-1} \circ u')^2(x)$, the last two of which are obtained from x via the (local) diffeomorphism $u^{-1} \circ u': M \supseteq U \cap U' \rightarrow U \cap U'$ which in turn is one of the two possible active interpretations of the change $u \mapsto u'$.¹⁵

4. Bundle view on the transformations laws of fields

The physical fields, classical and quantum ones, can be represented as sections of vector bundles. Such a view-point brings an additional light on their transformation properties.

4.1. Changes of frames in the bundle space

Suppose a physical field is described as a section $\varphi: M \rightarrow E$ of a vector bundle (E, π, M) . Here M is a real differentiable (4-)manifold (of class at least C^1), serving as a spacetime model, E is the bundle space and $\pi: M \rightarrow E$ is the projection; the fibres $\pi^{-1}(x)$, $x \in M$, are isomorphic vector spaces.¹⁶ Let (U, u) be a chart of M and $\{e_i\}$ be a (vector) frame in the bundle with domain containing U , i.e. $e_i: x \mapsto e_i(x) \in \pi^{-1}(x)$ with x in the domain of $\{e_i\}$ and $\{e_i(x)\}$ being a basis in $\pi^{-1}(x)$. Below we assume $x \in U \subseteq M$. Thus, similarly to (2.1), we have

$$\varphi: M \ni x \mapsto \varphi(x) = \varphi^i(x)e_i(x) = \varphi_u^i(\mathbf{x})e_i(u^{-1}(\mathbf{x})), \quad (4.1)$$

where

$$\mathbf{x} := u(x) \quad \varphi_u^i := \varphi^i \circ u^{-1} \quad (4.2)$$

and $\varphi^i(x)$ are the components of the vector $\varphi(x) \in \pi^{-1}(x)$ relative to the basis $\{e_i(x)\}$ in $\pi^{-1}(x)$.

Similarly, if (U', u') and $\{e'_i\}$ are other chart and frame, respectively, and $x \in U'$, then

$$\varphi: M \ni x \mapsto \varphi(x) = \varphi'^i(x)e'_i(x) = \varphi'^i_u(\mathbf{x}')e'_i(u'^{-1}(\mathbf{x}')), \quad (4.3)$$

where

$$\mathbf{x}' := u'(x) \quad \varphi'^i_u := \varphi'^i \circ u'^{-1}. \quad (4.4)$$

Further we shall suppose that $x \in U \cap U' \neq \emptyset$.

We can write the expressions

$$e'_i(x) = A^j_i(x)e_j(x) \quad (4.5)$$

$$e_i(x) = (A^{-1}(x))^j_i e'_j(x) \quad (4.5')$$

where $A: U \cap U' \rightarrow \text{GL}(\dim \pi^{-1}(x), \mathbb{R})$, $\text{GL}(n, \mathbb{K})$ being the general linear group of $n \times n$, $n \in \mathbb{N}$, matrices over \mathbb{K} , is the matrix-valued function defining the change $\{e_i\} \mapsto \{e'_i\}$. Combining these expressions with (4.1) and (4.3), we get

$$\varphi_u^i(\mathbf{x}) = A^i_j(x)\varphi'^j_u(\mathbf{x}') \quad (4.6)$$

$$\varphi'^i_u(\mathbf{x}') = (A^{-1}(x))^i_j \varphi_u^j(\mathbf{x}). \quad (4.6')$$

¹⁵ The another diffeomorphism is $u'^{-1} \circ u: M \supseteq U \rightarrow U' \subseteq M$, which is the inverse of $u^{-1} \circ u'$ on $U \cap U'$.

¹⁶ To make a contact with section 2, one should identify V with the (standard) fibre of (E, π, M) and consider $\varphi(x)$ as an element of $\pi^{-1}(x)$ rather than of V . Besides, now the Latin indexes refer to the bundle space and run from 1 to the fibre dimension of the bundle.

Generally the matrix $A(x)$ (resp. $A^{-1}(x)$) depends on the frames $\{e_i\}$ and $\{e'_i\}$ and describes the transformation $\{e_i\} \mapsto \{e'_i\}$ (resp. $\{e'_i\} \mapsto \{e_i\}$). One may reflect this by writing $A(\{e_i\}, \{e'_i\}; x)$ instead of $A(x)$.¹⁷ At this point we can make a connection with physics, which will make the above considerations more specific.

4.2. Transformations induced by coordinate changes

The physical concept of a *frame of reference* (or *reference frame*) is quite complex. However, for the purposes of the present work, it reduces to a collection (ordered pair) (u, e) of a coordinates system $u = (u^0, u^1, u^2, u^3)$, associated with a chart (U, u) ¹⁸ and a frame $e = \{e_i\}$ in E with domain containing U . The knowledge of (u, e) gives us the possibility to determine the coordinates of spacetime points (via u) and the components of the fields (via e).

It is a general opinion that the frames (and coordinates) in the bundle space E are not directly accessible for physical measurements.¹⁹ For this reason it is accepted that to any coordinate system u (or a chart (U, u)) there corresponds a unique frame $\{e_i\}$ in E in which the components of the physical fields φ are determined. The mapping $u \mapsto e$ seems to be unknown in the general case with exception of the tangent bundle case, $E = T(M)$, in which it is assumed $\{u^\mu\} \mapsto \{\frac{\partial}{\partial u^\mu}\}$, as well as its generalization to arbitrary tensor bundles over M . This is confirmed by the opinion that a change $u \mapsto u'$ implies transformation $e \mapsto e'$ when one changes the frame of reference from (u, e) to (u', e') . Thus, if (u, e) is a reference frame (regardless of is e induced by u) and we make a change $(u, e) \mapsto (u', e')$, then u' can be arbitrary (admissible) coordinate system and the change $u \mapsto u'$ completely determines the transformation $e \mapsto e'$.

As a consequence of the above (non-rigorous) motivation, we assume that, if (u, e) is a reference frame, then the transformation $u \mapsto u'$ implies a change $(u, e) \mapsto (u', e')$. where e' is given via (4.5) with

$$A^{-1}(x) \equiv A^{-1}(e, e'; x) = I(u \mapsto u'; x) = I^{-1}(u' \mapsto u; x) := (I(u' \mapsto u; x))^{-1} \quad (4.7)$$

which means that

$$e'_i(x) = (I^{-1}(u \mapsto u'; x))_i^j e_j(x) = I_i^j(u' \mapsto u; x) e_j(x) \quad (4.8)$$

$$e_i(x) = I_i^j(u \mapsto u'; x) e'_j(x), \quad (4.8')$$

where the matrix-valued function $I(u \mapsto u'; \cdot): U \cup U' \rightarrow \text{GL}(\dim \pi^{-1}(x), \mathbb{R})$ depends only on the change $u \mapsto u'$. Respectively now (4.6) and (4.6') read:

$$\varphi_u^i(\mathbf{x}) = (I^{-1}(u \mapsto u'; x))_j^i \varphi_{u'}^j(\mathbf{x}) \quad (4.9)$$

$$\varphi_{u'}^i(\mathbf{x}') = I_j^i(u \mapsto u'; x) \varphi_u^j(\mathbf{x}). \quad (4.9')$$

If the (admissible) transformations $u \mapsto u'$ form (a representation of) a group G , then the matrix-valued functions I form a representation of G .

¹⁷ If $\{e''_i\}$ is a third frame, the following relations hold:

$$\begin{aligned} A(\{e_i\}, \{e'_i\}; x) A(\{e'_i\}, \{e''_i\}; x) &= A(\{e_i\}, \{e''_i\}; x) \\ A(\{e_i\}, \{e'_i\}; x) &= A^{-1}(\{e'_i\}, \{e_i\}; x) \\ A(\{e_i\}, \{e_i\}; x) &= \mathbb{1} \end{aligned}$$

with $\mathbb{1}$ being the identity matrix. They express the simple fact that the transformations between bases or frames form a representation of the general linear group.

¹⁸ We identify $u: U \rightarrow \mathbb{R}^{\dim M}$ with $(u^0, \dots, u^{\dim M}): U \ni x \mapsto (u^0(x), \dots, u^{\dim M}(x)) \equiv u(x)$.

¹⁹ The only known possible exception being known (in a slightly modified form) as the Aharonov-Bohm effect [6–8], but there are some doubts in its reality.

4.3. Some peculiarities of the quantum fields

From fibre bundle point of view, the collection $\{\varphi_i\}$ of the (non smeared) components of a quantum field φ in a given reference frame is regarded as the set of the components of a section φ of certain vector fibre bundle (E, π, M) over a manifold M identified with the Minkowski spacetime. This means that in a frame $\{e^i\}$ in E over M ²⁰, we have

$$\text{Sec}(E, \pi, M) \ni \varphi: M \ni x \mapsto \varphi(x) = \varphi_i(x)e^i(x) \in \pi^{-1}(x). \quad (4.10)$$

In this setting, the components f^i of a vectorial test function \mathbf{f} should also be regarded as components of a section f of some vector bundle (F, π_F, M) such that in some frame $\{e_i\}$ in F over M is fulfilled

$$\text{Sec}(F, \pi_F, M) \ni f: M \ni x \mapsto f(x) = f^i(x)e_i(x) \in \pi_F^{-1}(x). \quad (4.11)$$

To retain the validity of (3.1), we should assume that (E, π, M) is the bundle dual to (F, π_F, M) , $(F, \pi_F, M)^* = (E, \pi, M)$. Hence the sections of (E, π, M) are, in fact, operator-valued linear mappings on the sections of (F, π_F, M) . Therefore, assuming $\{e^i\}$ to be the frame dual to $\{e_i\}$, i.e. $e^i = (e_i)^*$ with $e^i(e_j) = \delta_j^i$ ($= 0$ for $i \neq j$ and $= 1$ for $i = j$), we obtain

$$\varphi: f \mapsto \varphi(f): M \ni x \mapsto (\varphi(f))(x) := \varphi(x)(f(x)) = \sum_i \varphi_i(x)(f^i(x)) \quad (4.12)$$

as a result of which equation (3.1) takes the form

$$\varphi(\mathbf{f}) = \int \varphi(f).$$

To give a rigorous meaning of this integral one needs the notion of integration on manifolds (see, for instance, [9, ch. IV], [10, chapters IV and VII] and [11, ch. VIII]). To bypass this point, we shall assume below that the charts (U, u) and (U', u') are such that $u(U) = u'(U') = \mathbb{R}^4$ and the Jacobian of the change $u \mapsto u'$ to be equal to one, $\frac{\partial(u' \circ u^{-1})(r)}{\partial r} = 1$. Due to these assumptions, we can set

$$\varphi(\mathbf{f}) = \int_{\mathbb{R}^4} (\varphi(f) \circ u^{-1})(r) d^4r$$

which expression is independent of u ; to prove the last statement one may write the last formula with u' for u and change the integration variable to $q = (u \circ u'^{-1})(r)$.

So, if we make the change

$$e^i \mapsto e'^i = A_j^i e^j \quad (4.13a)$$

with a non-degenerate matrix-valued function $A = [A_i^j]$, then (cf. (4.5))

$$\varphi_i(x) \mapsto \varphi'_i(x) = (A^{-1})_i^j(x) \varphi_j(x) \quad (4.13b)$$

$$f^i(x) \mapsto f'^i(x) = A_j^i(x) f^j(x) \quad (4.13c)$$

$$e_i(x) \mapsto e'_i(x) = (A^{-1})_i^j(x) e_j(x). \quad (4.13d)$$

If $(u, \{e^j\})$, $u = (u^0, \dots, u^3)$ being a coordinate system, is a reference frame and we make a change $(u, \{e^j\}) \mapsto (u', \{e'^j\})$, then the components

$$\varphi_{u,i} := \varphi_i \circ u^{-1} \quad f_u^i := f^i \circ u^{-1} \quad (4.14)$$

²⁰ For technical reasons we label the basic vector fields with superscripts, i.e. we write e^i instead of e_i ; respectively, below the quantum fields are labeled via subscripts.

of respectively φ and f transform into (cf. (4.6'))

$$\varphi'_{u',i}(\mathbf{x}') = (A^{-1}(x))_i^j \varphi_{u,j}(\mathbf{x}) \quad (4.15a)$$

$$f'^i_{u'}(\mathbf{x}') = A^i_j(x) f^j_u(\mathbf{x}), \quad (4.15b)$$

where $\mathbf{x} := (x^0, \dots, x^3) = u(x) \in \mathbb{R}$ are the coordinates of $x \in M$ relative to u and similarly for \mathbf{x}' . In particular, we have to put in the above formulae

$$A^{-1}(x) = I(u \mapsto u'; x) \quad (4.16)$$

in a case when the change $u \mapsto u'$ induces $\{e^i\} \mapsto \{e'^i\}$.

Since the arguments leading to (3.5) are completely valid in the framework of fibre bundle approach, we can claim the existence of an operator U on the system's space of states such that

$$\varphi'_i(x) = U \circ \varphi_i(x) \circ U^{-1} \quad (4.17)$$

or, equivalently,

$$\varphi'_{u',i}(\mathbf{x}) = U \circ \varphi_{u,i}(\mathbf{x}) \circ U^{-1}. \quad (4.18)$$

Consequently, now the analogues of (3.19) read

$$\varphi'_i(x) = U \circ \varphi_i(x) \circ U^{-1} = (A^{-1}(x))_i^j \varphi_j(x) \quad (4.19)$$

$$\varphi'_{u',i}(\mathbf{x}) = U \circ \varphi_{u,i}(\mathbf{x}) \circ U^{-1} = (A^{-1}(x))_i^j \varphi_{u,j}(\mathbf{x}). \quad (4.20)$$

It should be noted, in all cases, quantum or classical ones, the set of admissible matrices $A(x)$ or $I(u \mapsto u'; x)$ defining the changes $\{e^i\} \mapsto \{e'^i\}$ generally does not coincide with the one of all non-degenerate matrices. Said differently, the set of admissible frames $\{e^i\}$ for a given field need not to be identical with the one of all frames. For instance, the set of admissible frames for a vector field, which is a section of the tangent bundle $(T(M), \pi_T, M)$, which are naturally associated with local coordinates, is the one of all coordinate frames (see (4.23b) below. Other example is a spinor field for which the matrix (4.16) has a quite special form (see (4.23c) below).²¹ What concerns scalar fields, for them the matrix $A(x)$ is the identity one, $A(x) = \mathbb{1}$.

4.4. Example: Relativistic quantum mechanics

Let M be the Minkowski spacetime. Suppose (M, u) is a global chart of M such that in the associated to it coordinate system $\{u^\mu\}$ the metric (tensor) has a Lorentzian form²² i.e., in physical terms, $\{u^\mu\}$ is (a part of) an inertial reference frame. The admissible (inertial) coordinate systems are obtained from one another via Poincaré transformations (2.5). So, if we set $x^\mu = u^\mu(x)$ for some $x \in M$ and $\mathbf{x} := (x^0, \dots, x^3)^\top$, then an admissible coordinate change can be represented as

$$\mathbf{x} \mapsto \mathbf{x}' = \Lambda \mathbf{x} + a, \quad (4.21)$$

where Λ is a 4×4 matrix representing 4-rotation and $a = (a^0, \dots, a^3)^\top$ is the collection of the components of a 4-vector representing a 4-translation.

We associate a matrix $I(u \mapsto u'; x)$ (see (4.7)) with the change (4.21) such that

$$I(u \mapsto u'; x) = I(\Lambda, x) \quad (4.22)$$

²¹ However, at the time being seems to be open the problem for finding a particular representative of the admissible frames (which will determine the remaining ones).

²² It is (often silently) accepted that the chart (M, id_M) and the associated to it Cartesian coordinate system have the property just described. This hypothesis should be included in the (physical) definition of the Minkowski spacetime.

depending on Λ , where the mapping $\Lambda \mapsto I(\Lambda, x)$ is a representation of the Poincaré group, and $x \in M$; the independence of a reflects the translational invariance of the theory.²³ The particular choice of the mapping $\Lambda \mapsto I(\Lambda, x)$ characterizes the particular field under considerations.²⁴ For example, we have:

$$I(\Lambda, x) = \mathbb{1} \quad \text{for spin-0 (scalar) field} \quad (4.23a)$$

$$(I(\Lambda, x))^\nu_\mu = \frac{\partial u'^\nu}{\partial u^\mu} \Big|_x = \Lambda^\nu_\mu \quad \text{for spin-1 (vector) field} \quad (4.23b)$$

$$I(\Lambda, x) = \exp\left(-\frac{i}{4}\omega\sigma_{\mu\nu}I_n^{\mu\nu}\right) \quad \text{for spin-}\frac{1}{2} \text{ (spinor) field,} \quad (4.23c)$$

where i is the imaginary unit, $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ is the commutator of the Dirac matrices γ_μ , Λ is a 4-rotation at an angle ω around an axis n , and I_n is the generator of this rotation.

Substituting (4.22) into (4.9'), we get

$$\varphi'_{u'}(\Lambda\mathbf{x} + a) = I_j^i(\Lambda, x)\varphi_u^j(\mathbf{x}) \quad (4.24)$$

or equivalently (see (2.8a))

$$\varphi'_{u'}(\mathbf{x}) = I_j^i(\Lambda, x)\varphi_u^j(\Lambda^{-1}(\mathbf{x} - a)). \quad (4.24')$$

4.5. Example: Quantum field theory

In quantum field theory the field components φ_i are known as field operators and are suppose to be operators (precisely, operator-valued distributions) acting on the Hilbert space of the system. In this case there exists a representation \mathcal{U} of the Poincaré group on the space of the stated vectors such that a coordinate change $\mathbf{x} \mapsto \mathbf{x}' = \Lambda\mathbf{x} + a$ induces the transformation (cf. (3.5))

$$\varphi_{u,i}(\mathbf{x}) \mapsto \varphi'_{u',i}(\mathbf{x}) = \mathcal{U}(\Lambda, a) \circ \varphi_{u,i}(\mathbf{x}) \circ \mathcal{U}^{-1}(\Lambda, a). \quad (4.25)$$

Thus, similarly to (3.15)–(3.17), we can write

$$\varphi'_{u',i}(\mathbf{x}) = I_i^j(\Lambda, x)\varphi_{u,j}(\Lambda^{-1}(\mathbf{x} - a)) \quad (4.26)$$

$$\varphi'_{u',i}(\mathbf{x}) = \mathcal{U}(\Lambda, a) \circ \varphi_{u,i}(\mathbf{x}) \circ \mathcal{U}^{-1}(\Lambda, a) \quad (4.27)$$

$$\mathcal{U}(\Lambda, a) \circ \varphi_{u,i}(\mathbf{x}) \circ \mathcal{U}^{-1}(\Lambda, a) = I_i^j(\Lambda; x)\varphi_{u,j}(\Lambda^{-1}(\mathbf{x} - a)) \quad (4.28)$$

which can be combined into

$$\varphi'_{u',i}(\mathbf{x}) = \mathcal{U}(\Lambda, a) \circ \varphi_{u,i}(\mathbf{x}) \circ \mathcal{U}^{-1}(\Lambda, a) = I_i^j(\Lambda; x)\varphi_{u,j}(\Lambda^{-1}(\mathbf{x} - a)) \quad (4.29)$$

where the particular form of the matrix $[I_i^j(\Lambda; x)]$ depends on the particular field under considerations (see, e.g., (4.23)).

In terms of the components φ_i of the section φ (see (4.10) and (4.14)) the relations (4.26)–(4.28) read

$$\varphi'_i(x) = I_i^j(\Lambda, x)\varphi_j(x) \quad (4.30)$$

$$\varphi'_i(x) = \mathcal{U}(\Lambda, a) \circ \varphi(x) \circ \mathcal{U}^{-1}(\Lambda, a) \quad (4.31)$$

$$\mathcal{U}(\Lambda, a) \circ \varphi(x) \circ \mathcal{U}^{-1}(\Lambda, a) = I_i^j(\Lambda, x)\varphi_j(x). \quad (4.32)$$

²³ In a case of local transformations, the matrix Λ may depend on the point $x \in M$.

²⁴ It also depends on the class of inertial reference frames as a whole. However, this class is fixed in our case via the requirement the coordinate system associated with the global chart (M, id_M) to be (a part of) an inertial reference frame.

According to (4.15b), (4.16) and (4.22), the test functions transform into

$$f'_{u'}{}^i(\Lambda \mathbf{x} + a) = (I^{-1}(\Lambda, x))_j^i f_u^j(\mathbf{x}) \quad (4.33)$$

or equivalently

$$f'_{u'}{}^i(\mathbf{x}) = (I^{-1}(\Lambda, x))_j^i f_u^j(\Lambda^{-1}(\mathbf{x} - a)) \quad (4.33')$$

which can also be written as (see (4.14), cf. (4.13c))

$$f'^i(\mathbf{x}) = (I^{-1}(\Lambda, x))_j^i f^j(x). \quad (4.34)$$

In this way, we see that in the fibre bundle approach both the classical and quantum fields transform according to passive coordinate transformations when one changes the frames of reference contrary to the conclusions at the end of section 3 in which fibre bundles were not involved.

We also observed that in the bundle approach for the derivation of equations like (4.26) and (4.33') one needs not to make additional hypotheses in contrast to the derivation of equalities like (3.12) and (3.8).

5. Heisenberg relations

As Heisenberg relations or equations in quantum field theory are known a kind of commutation relations between the field operators and the generators (of a representation) of a group acting on system's Hilbert space of states. Their (global) origin is in the equations (3.5) (or, equivalently, (3.4) or (3.2) and (3.3)) in which it is expected that the transformed field operators φ'_i can be expressed explicitly by means of φ_i via equations like (3.12). If the elements U (of the representation) of the group are labeled by $b = (b^1, \dots, b^s) \in \mathbb{K}^s$ for some $s \in \mathbb{N}$ (we are dealing, in fact, with a Lie group), i.e. we may write $U(b)$ for U , then the corresponding Heisenberg relations are obtained from (3.5) with $U(b)$ for U by differentiating it with respect to b^ω , $\omega = 1, \dots, s$, and then setting $b = b_0$, where $b_0 \in \mathbb{K}^s$ is such that $U(b_0)$ is the identity element.

The above shows that the Heisenberg relations are from pure geometric-group-theoretical origin and the only physics in them is the motivation leading to (3.2). However, there are strong evidences that to the Heisenberg relations can be given dynamical/physical sense by identifying/replacing in them the generators (of the representation) of the group by the corresponding operators of conserved physical quantities if the system considered is invariant with respect to this group (see, e.g. the discussion in [5, § 68]).

In subsections 5.1-5.3, we consider Heisenberg relations in the non-bundle approach (see sections 2 and 3), while in subsection 5.4 they are investigated on the ground of fibre bundles.

5.1. The Poincaré group

Suppose we study a quantum field with components φ_i relative to two reference frames connected by a general Poincaré transformation (2.5). Then the "global" version of the Heisenberg relations is expressed by the second equality in (3.19), i.e.

$$U(\Lambda, a) \circ \varphi_i(x) \circ U^{-1}(\Lambda, a) = D_i^j(\Lambda, a) \varphi_j(\Lambda x + a), \quad (5.1)$$

where U (resp. D) is a representation of the Poincaré group on the space of state vectors (resp. on the space of field operators), $U(\Lambda, a)$ (resp. $D(\Lambda, a) = [D_i^j(\Lambda, a)]$) is the mapping (resp. the matrix of the mapping) corresponding via U (resp. D) to (2.5), the index u in $\varphi_{u,i}$

is omitted (i.e. we write φ_i for $\varphi_{u,i}$) and the point $x \in M$ is identified with $\mathbf{x} = u(x) \in \mathbb{R}^4$. Since for $\Lambda = \mathbb{1}$ and $a = \mathbf{0} \in \mathbb{R}^4$ is fulfilled $u'(x) = u(x)$, we have

$$U(\mathbb{1}, \mathbf{0}) = \text{id} \quad D(\mathbb{1}, \mathbf{0}) = \mathbb{1}, \quad (5.2)$$

where id is the corresponding identity mapping and $\mathbb{1}$ stands for the corresponding identity matrix. Let $\Lambda = [\Lambda^\mu_\nu]$, $\Lambda^{\mu\nu} := \eta^{\nu\lambda} \Lambda^\mu_\lambda$, with $\eta^{\mu\nu}$ being the components of the Lorentzian metric with signature $(-+++)$, and define

$$T_\mu := \left. \frac{\partial U(\Lambda, a)}{\partial a^\mu} \right|_{(\Lambda, a) = (\mathbb{1}, \mathbf{0})} \quad (5.3a)$$

$$S_{\mu\nu} := \left. \frac{\partial U(\Lambda, a)}{\partial \Lambda^{\mu\nu}} \right|_{(\Lambda, a) = (\mathbb{1}, \mathbf{0})} \quad (5.3b)$$

$$H^i_{j\mu} := \left. \frac{\partial D^i_j(\Lambda, a)}{\partial a^\mu} \right|_{(\Lambda, a) = (\mathbb{1}, \mathbf{0})} \quad (5.3c)$$

$$I^i_{j\mu\nu} := \left. \frac{\partial D^i_j(\Lambda, a)}{\partial \Lambda^{\mu\nu}} \right|_{(\Lambda, a) = (\mathbb{1}, \mathbf{0})}. \quad (5.3d)$$

The particular form of the numbers $I^i_{j\mu\nu}$ depends on the field under consideration. In particular, we have (see (4.23))

$$I^1_{1\mu\nu} = 0 \quad \text{for spin-0 (scalar) field} \quad (5.4a)$$

$$I^\sigma_{\rho\mu\nu} = \delta^\sigma_\mu \eta_{\nu\rho} - \delta^\sigma_\nu \eta_{\mu\rho} \quad \text{for spin-1 (vector) field} \quad (5.4b)$$

$$[I^i_{j\mu\nu}]^4_{i,j=1} = -\frac{1}{2} i \sigma_{\mu\nu} \quad \text{for spin-}\frac{1}{2} \text{ (spinor) field.} \quad (5.4c)$$

Differentiating (5.1) relative to a^μ and setting after that $(\Lambda, a) = (\mathbb{1}, \mathbf{0})$, we find

$$[T_\mu, \varphi_i(x)]_- = \partial_\mu \varphi_i(x) + H^j_{i\mu} \varphi_j(x), \quad (5.5)$$

where $[A, B]_- := AB - BA$ is the commutator of some operators or matrices A and B . Since the field theories considered at the time being are invariant relative to spacetime translation of the coordinates, i.e. with respect to $\mathbf{x} \mapsto \mathbf{x} + a$, further we shall suppose that (cf. (5.3c))

$$H^i_{j\mu} = 0. \quad (5.6)$$

In this case (5.5) reduces to

$$[T_\mu, \varphi_i(x)]_- = \partial_\mu \varphi_i(x). \quad (5.7a)$$

Similarly, differentiation (5.1) with respect to $\Lambda^{\mu\nu}$ and putting after that $(\Lambda, a) = (\mathbb{1}, \mathbf{0})$, we obtain

$$[S_{\mu\nu}, \varphi_i(x)]_- = x_\mu \partial_\nu \varphi_i(x) - x_\nu \partial_\mu \varphi_i(x) + I^j_{i\mu\nu} \varphi_j(x) \quad (5.7b)$$

where $x_\mu := \eta_{\mu\nu} x^\nu$. The equations (5.7) are identical up to notation with [5, eqs.(11.70) and (11.73)]. Note that for complete correctness one should write $\varphi_{u,i}(\mathbf{x})$ instead of $\varphi_i(x)$ in (5.7), but we do not do this to keep our results near to the ones accepted in the physical literature [12–14].

As we have mentioned earlier, the particular Heisenberg relations (5.7) are from pure geometrical-group-theoretical origin. The following heuristic remark can give a dynamical sense to them. Recalling that the translation (resp. rotation) invariance of a (Lagrangian) field theory results in the conservation of system's momentum (resp. angular momentum) operator P_μ (resp. $M_{\mu\nu}$) and the correspondences

$$i\hbar T_\mu \mapsto P_\mu \quad i\hbar S_{\mu\nu} \mapsto M_{\mu\nu}, \quad (5.8)$$

with \hbar being the Planck's constant (divided by 2π), one may suppose the validity of the Heisenberg relations

$$[P_\mu, \varphi_i(x)]_- = i\hbar \partial_\mu \varphi_i(x) \quad (5.9a)$$

$$[M_{\mu\nu}, \varphi_i(x)]_- = i\hbar \{x_\mu \partial_\nu \varphi_i(x) - x_\nu \partial_\mu \varphi_i(x) + I_{i\mu\nu}^j \varphi_j(x)\}. \quad (5.9b)$$

However, one should be careful when applying the last two equations in the Lagrangian formalism as they are external to it and need a particular proof in this approach; e.g. they hold in the free field theory [13,15], but a general proof seems to be missing. In the axiomatic quantum field theory [3,4,12] these equations are identically valid as in it the generators of the translations (rotations) are identified up to a constant factor with the components of the (angular) momentum operator, $P_\mu = i\hbar T_\mu$ ($M_{\mu\nu} = i\hbar S_{\mu\nu}$).

5.2. Internal transformations

In our context, an internal transformation is a change of the reference frame such that the spacetime coordinates remain unchanged.

Let G be a group whose elements g_b are labeled by $b \in \mathbb{K}^s$ for some $s \in \mathbb{N}$.²⁵ Consider two reference frames $(u, \{e^i\})$ and $(u, \{e'^i\})$ (see section 2) connected via (2.3) with

$$A^{-1}(x) = I(b) \quad (5.10)$$

where $I: G \mapsto \text{GL}(\dim V, \mathbb{K})$ is a matrix representation of G and $I: G \ni g_b \mapsto I(b) \in \text{GL}(\dim V, \mathbb{K})$. Notice, here and below we label the elements of a frame with *superscripts* instead with subscripts, as in section 2. As a result of (3.5), (3.6) and (3.7) the field operators transform into

$$\varphi'_{u,i}(r) = U(b) \circ \varphi_{u,i}(r) \circ U^{-1}(b) \quad (5.11)$$

where U is a representation of G on the Hilbert space of state vectors and $U: G \ni g_b \mapsto U(b)$. Thus (3.17) now reduces to

$$U(b) \circ \varphi_{u,i}(r) \circ U^{-1}(b) = I_i^j(b) \varphi_{u,j}(r) \quad (5.12)$$

due to $u' = u$ in the case of internal transformations considered here.

Suppose $b_0 \in \mathbb{K}^s$ is such that g_{b_0} is the identity element of G and define

$$Q_\omega := \left. \frac{\partial U(b)}{\partial b^\omega} \right|_{b=b_0} \quad I_{i\omega}^j := \left. \frac{\partial I_i^j(b)}{\partial b^\omega} \right|_{b=b_0} \quad (5.13)$$

where $b = (b^1, \dots, b^s)$ and $\omega = 1, \dots, s$. Then, differentiating (5.12) with respect to b^ω and putting in the result $b = b_0$, we get the following Heisenberg relation

$$[Q_\omega, \varphi_{u,i}(r)]_- = I_{i\omega}^j \varphi_{u,j}(r) \quad (5.14)$$

or, if we identify $x \in M$ with $r = u(x)$ and omit the subscript u ,

$$[Q_\omega, \varphi_i(x)]_- = I_{i\omega}^j \varphi_j(x). \quad (5.15)$$

To make the situation more familiar, consider the case of one-dimensional group G , $s = 1$, when $\omega = 1$ due to which we shall identify b^1 with $b = (b^1)$. Besides, let us suppose that

$$I(b) = \mathbb{1} \exp(f(b) - f(b_0)) \quad (5.16)$$

²⁵ In fact, we are dealing with an s -dimensional Lie group and $b \in \mathbb{K}^s$ are the (local) coordinates of g_b in some chart on G containing g_{b_0} in its domain.

for some C^1 function f . Then (5.15) reduces to

$$[Q_1, \varphi_i(x)]_- = f'(b_0)\varphi_i(x), \quad (5.17)$$

where $f'(b) := \frac{df(b)}{db}$. In particular, if we are dealing with phase transformations, i.e.

$$U(b) = e^{\frac{1}{ie}bQ_1} \quad I(b) = \mathbb{1}e^{-\frac{q}{ie}b} \quad b \in \mathbb{R} \quad (5.18)$$

for some constants q and e (having a meaning of charge and unit charge, respectively) and operator Q_1 on system's Hilbert space of states (having a meaning of a charge operator), then (3.18) and (5.17) take the familiar form [12, eqs. (2.81) and (2-80)]

$$\varphi'_i(x) = e^{\frac{1}{ie}bQ_1} \circ \varphi_i(x) \circ e^{-\frac{1}{ie}bQ_1} = e^{-\frac{q}{ie}b}\varphi_i(x) \quad (5.19)$$

$$[Q_1, \varphi_i(x)]_- = -q\varphi_i(x). \quad (5.20)$$

The considerations in the framework of Lagrangian formalism invariant under phase transformations [12–14] implies conservation of the charge operator Q and suggests the correspondence (cf. (5.8))

$$Q_1 \mapsto Q \quad (5.21)$$

which in turn suggests the Heisenberg relation

$$[Q, \varphi_i(x)]_- = -q\varphi_i(x). \quad (5.22)$$

We should note that this equation is external to the Lagrangian formalism and requires a proof in it [15].

5.3. The general case

As we saw in the previous subsections, the corner stone of the (global) Heisenberg relations is the equation (3.17),

$$U \circ \varphi_{u,i}(r) \circ U^{-1} = \frac{\partial(u' \circ u^{-1})(r)}{\partial r} (A^{-1}(u^{-1}(r)))^j_i \varphi_{u,j}((u' \circ u^{-1})(r)). \quad (5.23)$$

Now, following the ideas at the beginning of this section, we shall demonstrate how from it can be derived Heisenberg relations in the general case.

Let G be an s -dimensional, $s \in \mathbb{N}$, Lie group. Without going into details, we admit that its elements are labeled by $b = (b^1, \dots, b^s) \in \mathbb{K}^s$ and g_{b_0} is the identity element of G for some fixed $b_0 \in \mathbb{K}^s$. Suppose that there are given three representations H , I and U of G and consider frames of reference with the following properties:

1. $H: G \ni g_b \mapsto H_b: \mathbb{R}^{\dim M} \rightarrow \mathbb{R}^{\dim M}$ and any change $(U, u) \mapsto (U', u')$ of the charts of M is such that $u' \circ u^{-1} = H_b$ for some $b \in \mathbb{K}^s$.
2. $I: G \ni g_b \mapsto I(b) \in \text{GL}(\dim V, \mathbb{K})$ and any change $\{e^i\} \mapsto \{e'^i = A^i_j e^j\}$ of the frames in V is such that $A^{-1}(x) = I(b)$ for all $x \in M$ and some $b \in \mathbb{K}^s$.
3. $U: G \ni g_b \mapsto U(b)$, where $U(b)$ is an operator on the space of state vectors, and a change $(u, \{e^i\}) \mapsto (u', \{e'^i\})$ of the reference frame entails (3.5) with $U(b)$ for U .

Under the above hypotheses equation (5.23) transforms into

$$U(b) \circ \varphi_{u,i}(r) \circ U^{-1}(b) = \det \left[\frac{\partial(H_b(r))^k}{\partial r^l} \right] I^j_i(b) \varphi_{u,j}(H_b(r)) \quad (5.24)$$

which can be called *global Heisenberg relation* in the particular situation. The next step is to differentiate this equation with respect to b^ω , $\omega = 1, \dots, s$, and then to put $b = b_0$ in the result. In this way we obtain the following (*local*) *Heisenberg relation*

$$[U_\omega, \varphi_{u,i}(r)]_- = \Delta_\omega(r) \varphi_{u,i}(r) + I_{i\omega}^j \varphi_{u,j}(r) + (h_\omega(r))^k \frac{\partial \varphi_{u,i}(r)}{\partial r^k}, \quad (5.25)$$

where

$$U_\omega := \left. \frac{\partial U(b)}{\partial b^\omega} \right|_{b=b_0} \quad (5.26a)$$

$$\Delta_\omega(r) := \left. \frac{\partial \det \left[\frac{\partial (H_b(r))^i}{\partial r^j} \right]}{\partial b^\omega} \right|_{b=b_0} \in \mathbb{K} \quad (5.26b)$$

$$I_{i\omega}^j := \left. \frac{\partial I_i^j(b)}{\partial b^\omega} \right|_{b=b_0} \in \mathbb{K} \quad (5.26c)$$

$$h_\omega := \left. \frac{\partial H_b}{\partial b^\omega} \right|_{b=b_0} : \mathbb{R}^{\dim M} \rightarrow \mathbb{R}^{\dim M}. \quad (5.26d)$$

In particular, if H_b is linear and non-homogeneous, i.e. $H_b(r) = H(b) \cdot r + a(b)$ for some $H(b) \in \text{GL}(\dim M, \mathbb{R})$ and $a(b) \in \mathbb{K}^{\dim M}$ with $H(b_0) = \mathbb{1}$ and $a(b_0) = \mathbf{0}$, then (Tr means trace of a matrix or operator)

$$\Delta_\omega(r) = \left. \frac{\partial \det(H(b))}{\partial b^\omega} \right|_{b=b_0} = \left. \frac{\partial \text{Tr}(H(b))}{\partial b^\omega} \right|_{b=b_0} \quad h_\omega(\cdot) = \left. \frac{\partial H(b)}{\partial b^\omega} \right|_{b=b_0} \cdot (\cdot) + \left. \frac{\partial a(b)}{\partial b^\omega} \right|_{b=b_0} \quad (5.27)$$

as $\left. \frac{\partial \det B}{\partial b_i^j} \right|_{B=\mathbb{1}} = \delta_j^i$ for any square matrix $B = [b_i^j]$. In this setting the Heisenberg relations corresponding to Poincaré transformations (see subsection 5.1) are described by $b \mapsto (\Lambda^{\mu\nu}, a^\lambda)$, $H(b) \mapsto \Lambda$, $a(b) \mapsto a$ and $I(b) \mapsto I(\Lambda)$, so that $U_\omega \mapsto (S_{\mu\nu}, T_\lambda)$, $\Delta_\omega(r) \equiv 0$, $I_{i\omega}^j \mapsto (I_{i\mu\nu}^j, 0)$ and $(h_\omega(r))^k \frac{\partial}{\partial r^k} \mapsto r_\mu \frac{\partial}{\partial r^\nu} - r_\nu \frac{\partial}{\partial r^\mu}$.

The case of internal transformations, considered in the previous subsection, corresponds to $H_b = \text{id}_{\mathbb{R}^{\dim M}}$ and, consequently, in it $\Delta_\omega(r) \equiv 0$ and $h_\omega = 0$.

5.4. Fibre bundle approach

The origin of the Heisenberg relations on the background of fibre bundle setting is in any one of the equivalent second equalities in (4.19) or (4.20),

$$U \circ \varphi_i(x) \circ U^{-1} = (A^{-1})_i^j(x) \varphi_j(x) \quad (5.28)$$

$$U \circ \varphi_{u,i}(x) \circ U^{-1} = (A^{-1})_i^j(x) \varphi_{u,j}(x). \quad (5.28')$$

Similarly to subsection 5.3, consider a Lie group G , its representations I and U and reference frames with the following properties:

1. $I: G \ni g_b \mapsto I(b) \in \text{GL}(\dim V, \mathbb{K})$ and the changes $\{e_i\} \mapsto \{e'^i = A_j^i e^j\}$ of the frames in V are such that $A^{-1}(x) = I(b)$ for all $x \in M$ and some $b \in \mathbb{K}^s$.
2. $U: g \ni g_b \mapsto U(b)$, where $U(b)$ is an operator on the space of state vectors, and the changes $(u, \{e^i\}) \mapsto (u', \{e'^i\})$ of the reference frames entail (3.5) with $U(b)$ for U .

Remark 5.1. One can consider also simultaneous coordinate changes $u \mapsto u' = H_b \circ u$ induced by a representation $H: G \ni g_b \mapsto H_b: \mathbb{R}^{\dim M} \rightarrow \mathbb{R}^{\dim M}$, as in subsection 5.3. However such a supposition does not influence our results as the basic equations (5.29) and (5.29') below are independent from it; in fact, equation (5.29) is coordinate-independent, while (5.29') is its version valid in any local chart (U, u) as $\varphi_u := \varphi \circ u^{-1}$ and $x := u(x)$.

Thus equations (5.28) and (5.28') transform into the following *global Heisenberg relations* (cf. (5.24))

$$U(b) \circ \varphi_i(x) \circ U^{-1}(b) = I_i^j(b) \varphi_j(x) \quad (5.29)$$

$$U(b) \circ \varphi_{u,i}(\mathbf{x}) \circ U^{-1}(b) = I_i^j(b) \varphi_{u,j}(\mathbf{x}). \quad (5.29')$$

Differentiating (5.29) with respect to b^ω and then putting $b = b_0$, we derive the following (*local*) *Heisenberg relations*

$$[U_\omega, \varphi_i(x)]_- = I_{i\omega}^j \varphi_j(x) \quad (5.30)$$

or its equivalent version (cf. (5.25))

$$[U_\omega, \varphi_{u,i}(\mathbf{x})]_- = I_{i\omega}^j \varphi_{u,j}(\mathbf{x}), \quad (5.30')$$

where

$$U_\omega := \left. \frac{\partial U(b)}{\partial b^\omega} \right|_{b=b_0} \quad (5.31a)$$

$$I_{i\omega}^j := \left. \frac{\partial I_i^j(b)}{\partial b^\omega} \right|_{b=b_0}. \quad (5.31b)$$

Recalling (4.10), we can rewire the Heisenberg relations obtained as

$$[U_\omega, \varphi]_- = I_{i\omega}^j \varphi_j e^i. \quad (5.32)$$

One can prove that the r.h.s. of this equation is independent of the particular frame $\{e^i\}$ in which it is represented.

The case of Poncaré transformations is described by the replacements $b \mapsto (\Lambda^{\mu\nu}, a^\lambda)$, $U_\omega \mapsto (S_{\mu\nu}, T_\lambda)$ and $I_{i\omega}^j \mapsto (I_{i\mu\nu}^j, 0)$ (for translation invariant theory) and, consequently, the equations (5.29) and (5.29') now read

$$U(\Lambda, a) \circ \varphi_i(x) \circ U^{-1}(\Lambda, a) = I_i^j(\Lambda, a) \varphi_j(x) \quad (5.33)$$

$$U(\Lambda, a) \circ \varphi_{u,i}(\mathbf{x}) \circ U^{-1}(\Lambda, a) = I_i^j(\Lambda, a) \varphi_{u,j}(\mathbf{x}). \quad (5.33')$$

Hence, for instance, the Heisenberg relation (5.30) now takes the form (cf. (5.7))

$$[T_\mu, \varphi_i(x)]_- = 0 \quad (5.34a)$$

$$[S_{\mu\nu}, \varphi_i(x)]_- = I_{i\mu\nu}^j \varphi_j(x). \quad (5.34b)$$

Respectively, the correspondences (5.8) transform these equations into

$$[P_\mu, \varphi_i(x)]_- = 0 \quad (5.35a)$$

$$[M_{\mu\nu}, \varphi_i(x)]_- = I_{i\mu\nu}^j \varphi_j(x) \quad (5.35b)$$

which now replace (5.9).

Since equation (5.9a) (and partially equation (5.9b)) is (are) the corner stone for the particle interpretation of quantum field theory [13–15], the equation (5.35a) (and partially equation (5.35b)) is (are) physically unacceptable if one wants to retain the particle interpretation in the fibre bundle approach to the theory. For this reason, it seems that the correspondences (5.8) should *not* be accepted in the fibre bundle approach to quantum field theory, in which (5.7) transform into (5.34). However, for retaining the particle interpretation one can impose (5.9) as subsidiary restrictions on the theory in the fibre bundle approach.

It is almost evident that this is possible if the frames used are connected by linear homogeneous transformations with spacetime constant matrices, $A(x) = \text{const}$ or $\partial_\mu A(x) = 0$. Consequently, if one wants to retain the particle interpretation of the theory, one should suppose the validity of (5.9) in some frame and, then, it will hold in the whole class of frames obtained from the chosen one by transformations with spacetime independent matrices.

Since the general setting investigated above is independent of any (local) coordinates, it describes also the fibre bundle version of the case of internal transformations considered in subsection 5.2. This explains why equations line (5.14) and (5.30') are identical but the meaning of the quantities $\varphi_{u,i}$ and $I_{i\omega}^j$ in them is different.²⁶ In particular, in the case of phase transformations

$$U(b) = e^{\frac{1}{ie}bQ_1} \quad I(b) = \mathbb{1}e^{-\frac{q}{ie}b} \quad b \in \mathbb{R} \quad (5.36)$$

the Heisenberg relations (5.30) reduce to

$$[Q_1, \varphi_i(x)]_- = -q\varphi_i(x), \quad (5.37)$$

which is identical with (5.20), but now φ_i are the components of the section φ in $\{e^i\}$. The invariant form of the last relations is

$$[Q_1, \varphi]_- = -q\varphi \quad (5.38)$$

which is also a consequence from (5.32) and (5.19).

6. Conclusion

In this paper we have shown how the transformation laws of the components of the classical and quantum fields arise.

We also have demonstrated how the Heisenberg equations can be derived in the general case and in particular situations. They are from pure geometrical origin and one should be careful when applying them in the Lagrangian formalism in which they are subsidiary conditions, like the Lorenz gauge in the quantum electrodynamics. In the general case they need not to be consistent with the Lagrangian formalism and their validity should carefully be checked. For instance, if one starts with field operators in the Lagrangian formalism of free fields and adds to it the Heisenberg relations (5.9a) concerning the momentum operator, then the arising scheme is not consistent as in it start to appear distributions, like the Dirac delta function. This conclusion leads to the consideration of the quantum fields as operator-valued distribution in the Lagrangian formalism even for free fields. In the last case, the Heisenberg relations concerning the momentum operator are consistent with the Lagrangian formalism. Besides, they play an important role in the particle interpretation of the so-arising theory.

Let us now pay some attention on the observability and measurability of the objects considered in this paper.

We say that an object is observable if we can obtain information from it directly or via its interaction with other objects(s) and we can detect (and interpret) the result(s) of this interaction. It is measurable if we can measure some its characteristics obtained in the process of observation by assigning to them numerical values. One should assume that the physical fields are observable and measurable as otherwise they cannot be studied rigorously. At this point arises the question: are the components φ_i of a field φ observable and possibly measurable? Since the set $\{\varphi_i\}$ is in a sense a projection of φ on a reference frame, this

²⁶ Note, now $I(b)$ is the matrix defining transformations of frames in the bundle space, while in (5.18) it serves a similar role for frames in the vector space V .

question is equivalent to the problem: are the reference frames observable and possibly measurable?

In the context of the present paper, a reference frame is a pair (u, e) of a coordinate system $u = (u^0, \dots, u^{\dim M-1})$ on a manifold M , representing a spacetime model, and a frame $e = (e^1, \dots, e^n)$ in a vector space V or in the bundle space E of a fibre bundle (E, π, M) with $n \in \mathbb{N}$ being the dimension of V or the fibre dimension of (E, π, M) .

It is known that (locally) there exist experimental procedures which (at least in principle) allow to be determined the coordinates $\mathbf{x} = (x^0, \dots, x^{\dim M-1})$ of a spacetime point $x \in M$ with respect to some set of real physical objects; in fact, this set is the object we mathematically describe via a coordinate system u which, in this setting, is defined by $u: x \mapsto u(x) := \mathbf{x}$. Thus the coordinates of a spacetime point are observable and measurable. From a coordinate system $\{u^\mu\}$ on M are construct the coordinate frames $\{\frac{\partial}{\partial u^\mu}\}$ and $\{du^\mu\}$ in respectively the tangent and cotangent bundles over M and from them, by tensor multiplication, can be constructed frames in the tensor bundles over M . Therefore we can claim that the coordinate frames in tensor bundles are (indirectly) observable and measurable. Since the elements of a non-coordinate frame in a tensor bundle can be represented as linear combinations (with functions as coefficients) of the ones of an arbitrarily chosen coordinate frame (in the overlap of their domains), we can also claim that the non-coordinate frames in tensor bundles are also observable and measurable. From here follows that if a tensor field is observable and measurable, then such are and its componets.

Let us turn now our attention to elements of the (total) bundle space of a fibre bundle, which is not a tensor bundle. Are the sections of such a bundle and/or their components observable and, possible, measurable? In particular, are the frames in non-tensor bundles observable and, possible, measurable? It seems that the answer to these questions are in general negative. For instance, the elements of a spinor bundle describing $(\text{spin-}\frac{1}{2})$ Dirac field are not observable but from them can be constructed observable quantities like the energy-momentum and charge characteristics of the field [13].

At the time being there is only one phenomenon, known as the Aharonov-Bohm effect [6–8], that may lead to observability of elements of a non-tensor bundle. Its essence is that the electromagnetic potential can give rise to directly observable results and, in this sense, are observable. Let us suppose that this is true ²⁷ As the electromagnetic potentials A_μ are coefficients of a linear connection ∇ in C^1 one-dimensional vector bundle (E, π, M) , we have $\nabla_{\frac{\partial}{\partial u^\mu}} e = A_\mu e$, where $\{u^\mu\}$ is a coordinate system on M and $\{e\}$ is a frame in E consisting of a single section $e: M \rightarrow E$ with non-zero values. A change $(\{u^\mu\}, e) \mapsto (\{u'^\mu\}, e' = fe)$ with $f: M \rightarrow \mathbb{R} \setminus \{0\}$ implies [16, p. 356, eq. (4.23)] $A_\mu \mapsto A'_\mu = \frac{\partial u^\nu}{\partial u'^\mu} (A_\nu + \frac{\partial \ln f}{\partial u^\nu})$; in particular, if $u'^\mu = u^\mu$, then we have a pure gauge transformation $A_\mu \mapsto A'_\mu = A_\mu + \frac{\partial \ln f}{\partial u^\mu}$. Therefore the frame $\{e\}$ is observable via the electromagnetic potentials A_μ in the reference frame $(\{u^\mu\}, \{e\})$. However, since a change of the frame $\{e\}$ is defined within a constant non-vanishing factor, which does not change A_μ , the frame $\{e\}$ is not measurable regardless of are the potentials A_μ measurable or not measurable.

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²⁷ There are some doubts in the reality of the Aharonov-Bohm effect.

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